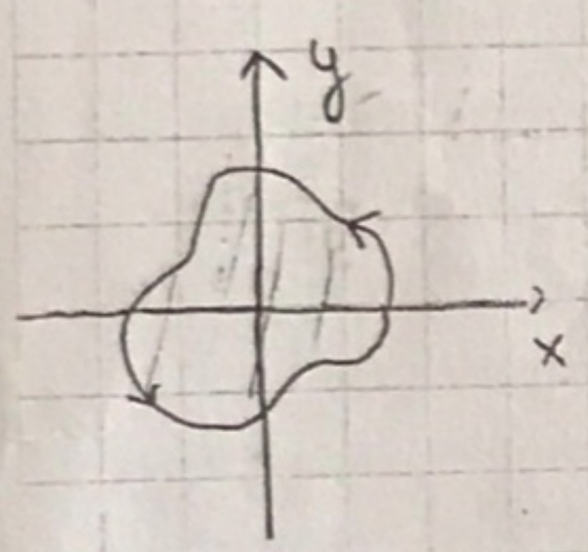
Formula Gaus - Ostrogradski:

$P, Q, R \in C^1(\bar{\Omega} \subseteq \mathbb{R}^3)$, Ω - ograničena oblast dvodimenzionalnom dio po dio glatkom zatvorenom površi M^2 $(\partial\Omega = M^2)$ na kojoj su f-je P, Q, R takođe C^1. Ako je M^2 orijentisana spoljnim normalama, onda važi:

$$\int_{\vec{M}^2} P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy = \iiint_{\Omega} \operatorname{div}(P, Q, R)\,dx \wedge dy \wedge dz$$

$$= \iiint_{\Omega} (P_x + Q_y + R_z)\,dx \wedge dy \wedge dz$$

1° Grin

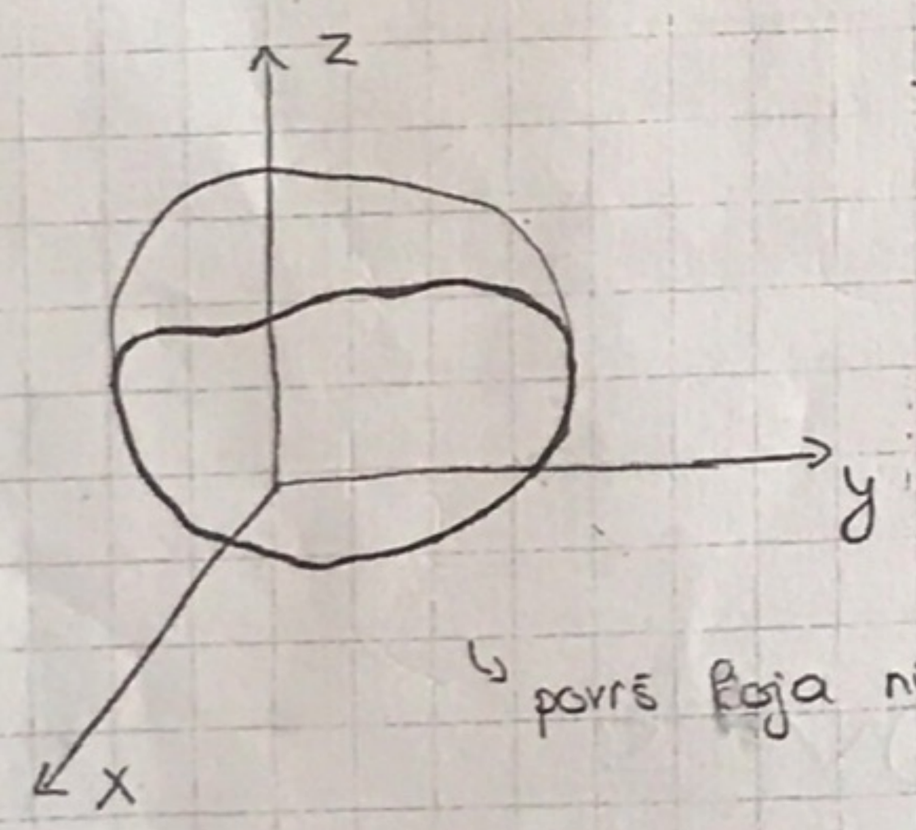


-> u ravni imamo neku krivu koja je zatvorena

-> pozitivna orijentacija

Računanje po krivoj možemo da prebacimo na računanje dvostrukog integrala.

2° Štoks



-> imamo krivu u $\mathbb{R}^3$ koja je zatvorena

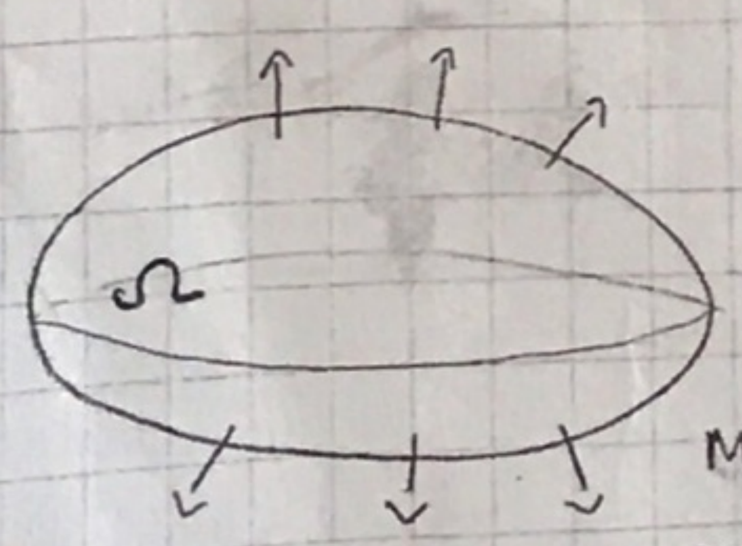
-> napravimo neku oblast nad njom (površ sa krajem) i izračunamo int. po toj površi

↳ površ koja nije skroz zatvorena

3° Gaus

-> preko divergencije $(\operatorname{div}(P, Q, R))$ prebacamo na računanje po unutrašnjosti

-> svede se na obični, trostruki int



M^2 - pozitivno orijentisana površ koja ograničava oblast Ω

↳ skroz zatvorena površ

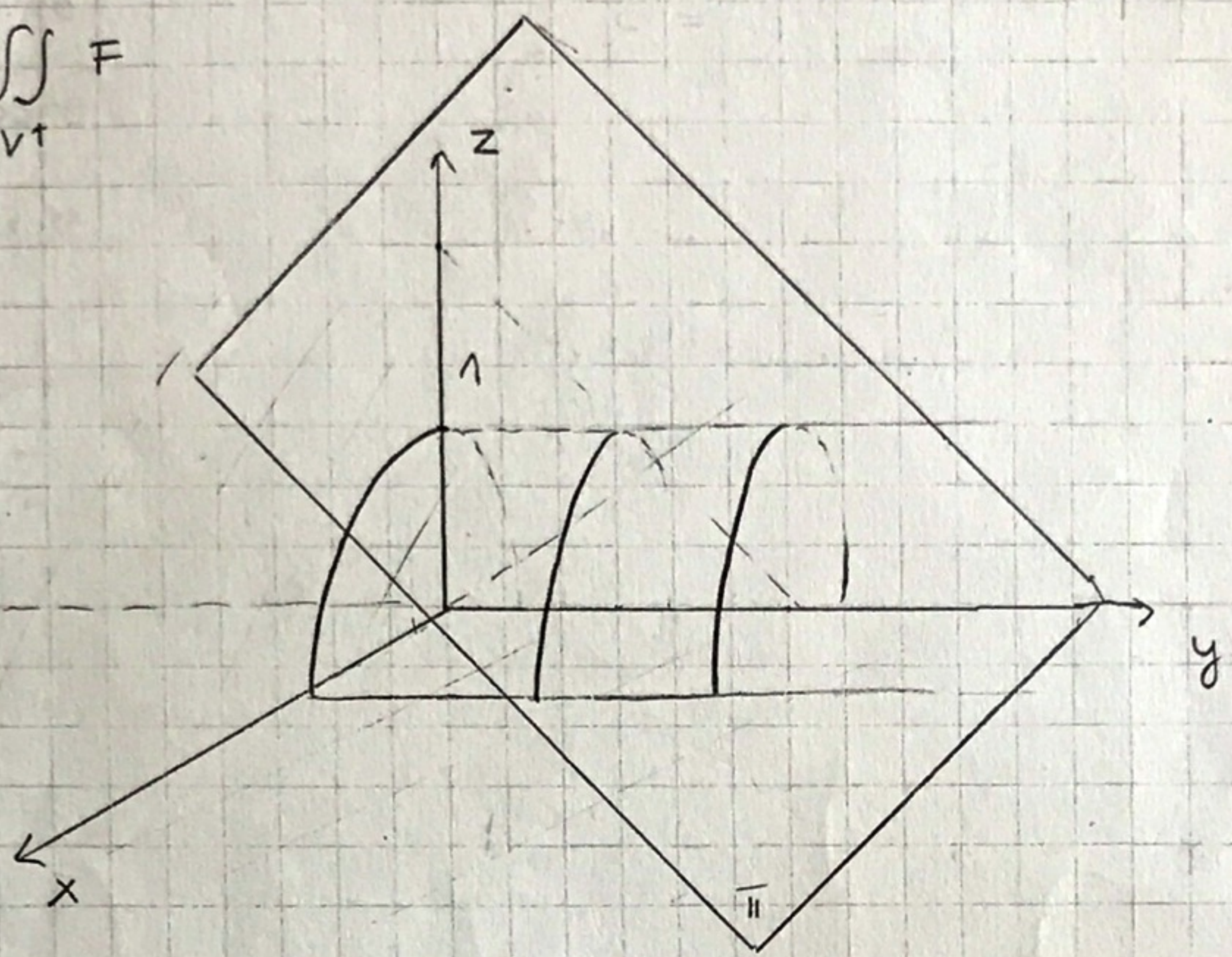
5. $F(x,y,z) = (xy,\ y^2 + e^{xz^2},\ \sin(xy))$

V - tijelo ograničeno ravnima $z=0$, $y=0$, $y+z=2$ i paraboloičnim cilindrom $z=1-x^2$.

Korisreći Gaus-Os, izračunati:

$$\iint_{\partial V^+} F$$

R



$P_x + Q_y + R_z = 3y$

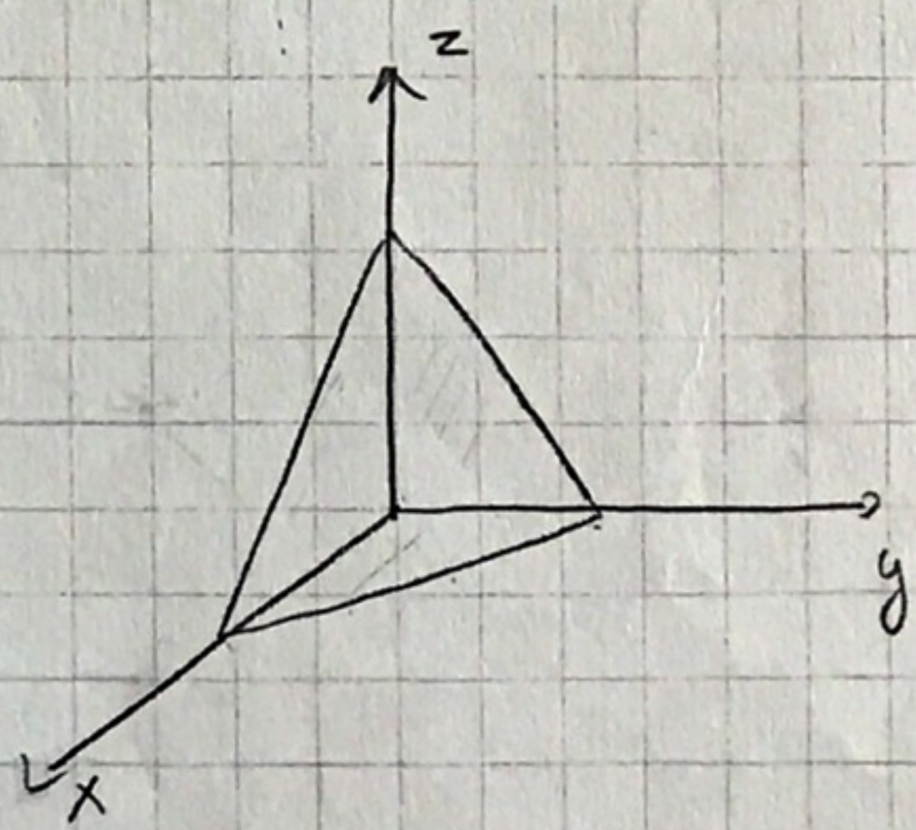
$$I = \int_{-1}^{1}\int_{0}^{1-x^2}\int_{0}^{2-z} 3y\, dy\, dz\, dx = \dots = \frac{184}{35}$$

6. $\iint_{S} x^2 dy dz + y^2 dz dx + z^2 dx dy$

S - spolja orjentisan tetraedar:

R

$$V: \begin{cases} x \geq 0 \\ y \geq 0 \\ z \geq 0 \\ x+y+z \leq 1 \end{cases}$$



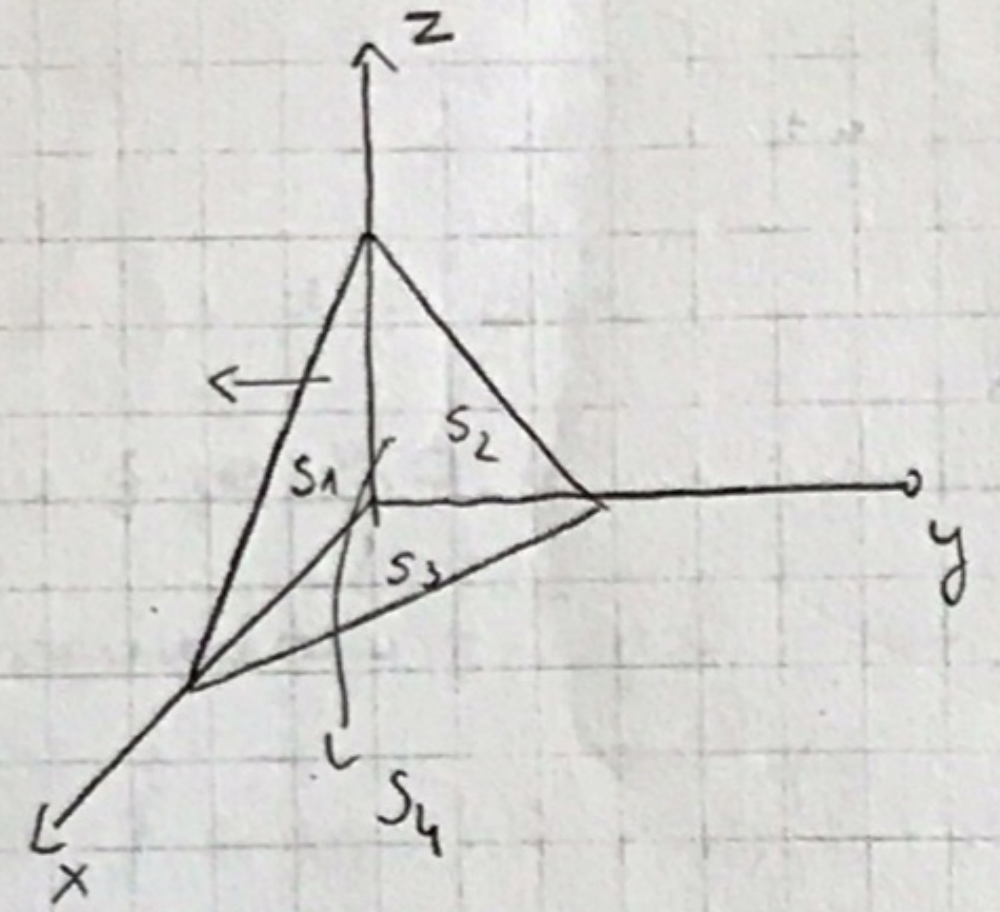
$\partial V = S$

I način: Gaus

$$\operatorname{div} F = 2x + 2y + 2z \Rightarrow \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} 2(x+y+z)\, dz = \dots = \frac{1}{4}$$

↳ integral po unutrašnjosti tetraedra

II način:



$S_1: (x,z) \to (x,0,z)$

$\vec{n} = (0,-1,0) = g_x \times g_z$

$\vec{g_x} \times g_z$

$I_{S_1} = 0$

S_2, S_3 isto

$\vec{n} = (-1,0,0)$ $\quad \vec{n}(0,0,-1)$

$I = 0$ $\quad I = 0$

$(x,y) \to (x,y,1-x-y)$

$S_4: \quad \vec{n} = (1,1,1)$

$$\int_0^1\int_0^{1-x}(x^2+y^2+(1-x-y)^2)\,dx\,dy = \dots = \frac{1}{4}$$

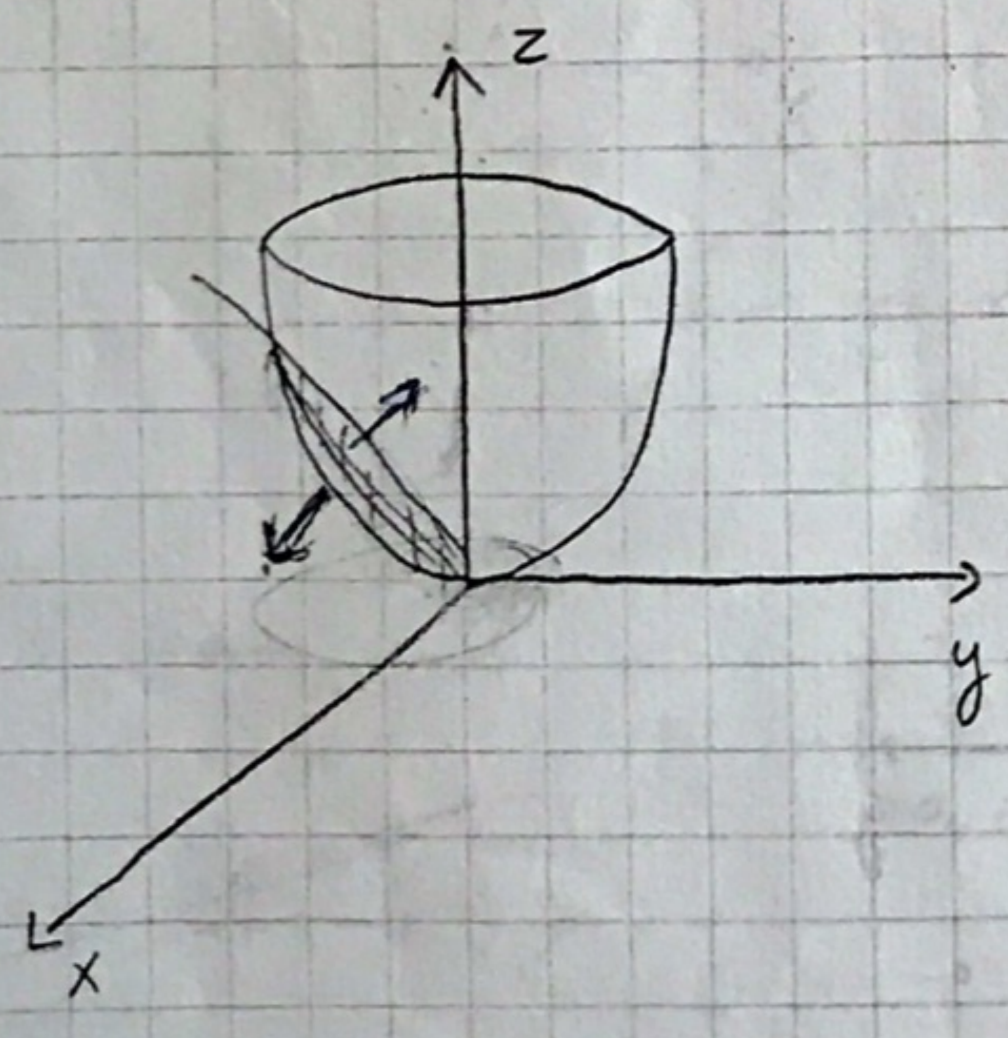
7. $\iint\limits_S x^3\,dy \wedge dz + y^3\,dz \wedge dx + z^2\,dx \wedge dy$

poz. orij.

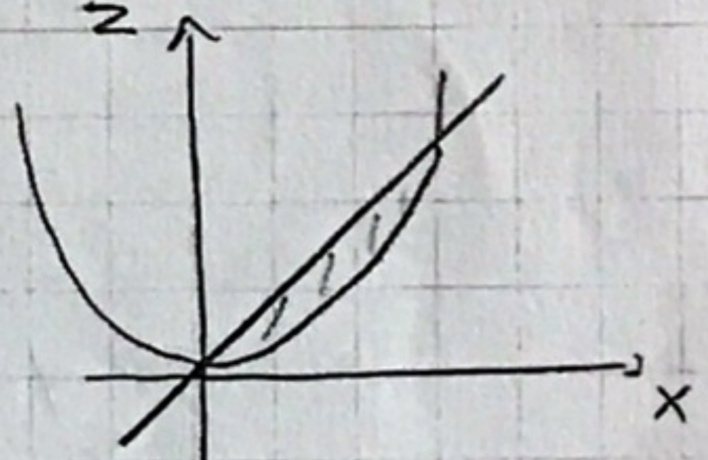
$z = x^2 + y^2 \quad (1)$

$z = 2x \quad (2)$

S je granica tijela ograničenog sa površima (1) i (2).



u xOz:



I način: Gaus-Ostrogradski

$z = 2x$ $\qquad$ $\operatorname{div} F = 3x^2 + 3y^2 + 2z$

$x^2 + y^2 = z$

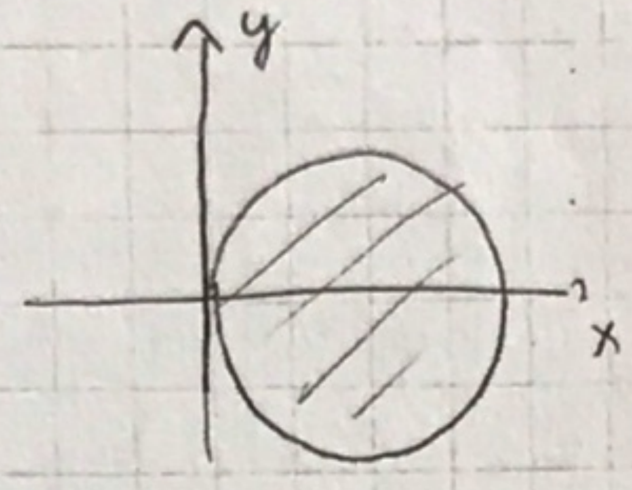
$(x-1)^2 + y^2 = 1$

→ uvodimo cil. koo.

$x = r\cos\varphi$

$y = r\sin\varphi$ $\qquad$ $z = z$

$J = r$



$\varphi \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$r = 2\cos\varphi$

$$\Rightarrow \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{0}^{2\cos\varphi} \int_{r^2}^{2r\cos\varphi} (3r^2 + 2z)\, r\, dz\, dr\, d\varphi = \dots = \frac{11\pi}{3}$$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^6\varphi = \frac{5\pi}{16}$$

$u = \cos^5(x)$ $\qquad$ $du = 5\cos^4(x)(-\sin x)$

$dv = \cos(x)$ $\qquad$ $v = \sin x$

II način:

$(x, y) \xrightarrow{g} (x, y, x^2 + y^2)$ - dio paraboloida

$g_x \times g_y = (-2x, -2y, 1)$

ugao tup pa $z < 0 \Rightarrow \varepsilon = -1$

$$= -\iint_D \left(x^3(-2x) + y^3(-2y) + (x^2+y^2)^2\right) dx\, dy = \dots = \frac{39\pi}{18}$$

$(x, y) \to (x, y, 2x)$ - ravan

$z = 2x \Rightarrow \vec{n}\,(-2, 0, 1)$

$\Updownarrow$

$2x - z = 0$ $\qquad$ $\varepsilon = 1$

$$= \iint_D \left(x^3(-2) + 4x^2 \cdot 1\right) dx\, dy = \dots = \frac{3\pi}{2}$$

$$\Rightarrow \frac{39\pi}{18} + \frac{3\pi}{2} = \frac{11\pi}{3}$$

8. $A = (x^2, y^2, z^2)$

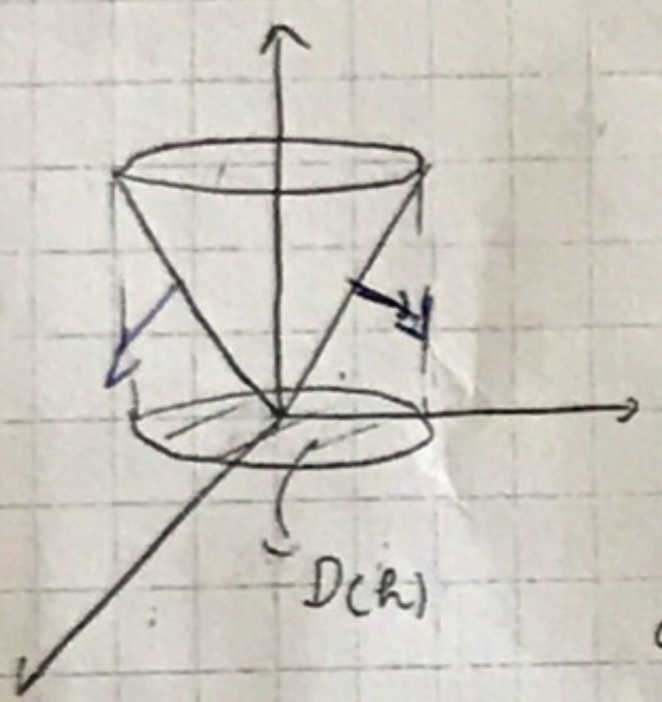
$$S: \begin{cases} z = \sqrt{x^2+y^2} \\ 0 \le z \le h \end{cases}$$

Na 2 načina izračunati $\iint\limits_{S} A = ?$ S → orjentisan po spoljnim $\underline{\underline{n}}$

R.

I način: direktno

$$(x,y) \to (x, y, \sqrt{x^2+y^2})$$



$$g_x = \left(1, 0, \frac{x}{\sqrt{x^2+y^2}}\right)$$

$$g_y = \left(0, 1, \frac{y}{\sqrt{\cdots}}\right)$$

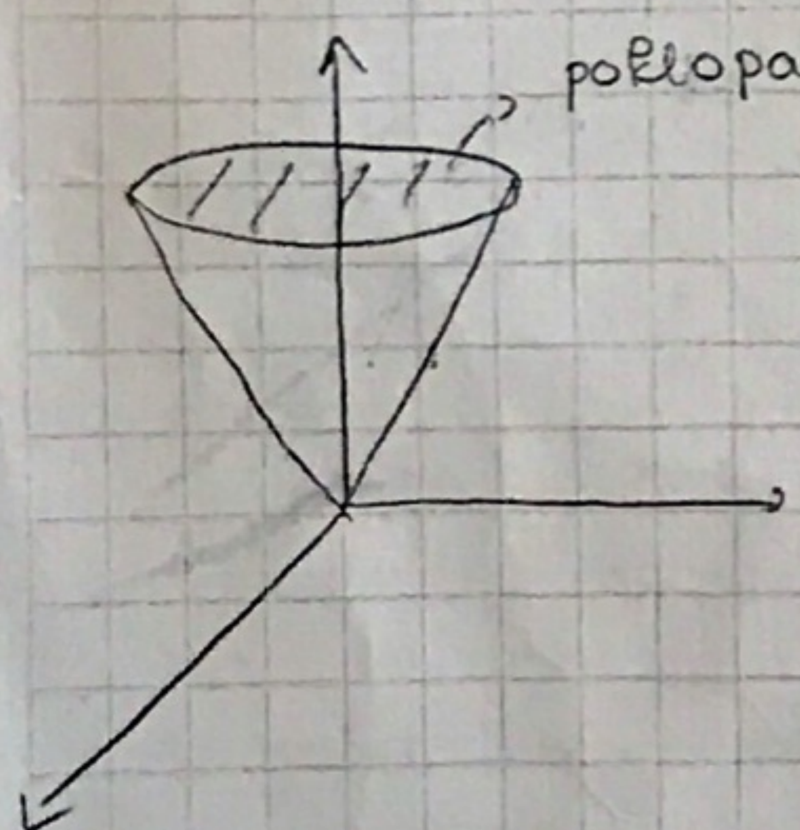
$$g_x \times g_y = \left(-\frac{x}{\sqrt{\cdots}}, \frac{-y}{\sqrt{\cdots}}, 1\right)$$

$$\varepsilon = -1$$

$$\Rightarrow \iint\limits_{D(h)} \Big(\underbrace{x^2 \cdot \frac{x}{\sqrt{1-x^2-y^2}} + y^2 \cdot \frac{y}{\sqrt{\cdots}}}_{=0} + (x^2+y^2)(-1)\Big)\, dx\, dy =$$

$$= -\iint\limits_{D(h)} (x^2+y^2)\, dx\, dy = -\int_0^h \int_0^{2\pi} r^2 \cdot r\, d\varphi\, dr = -2\pi \cdot \frac{h^4}{4} = -\frac{\pi h^4}{2}$$

II način: (Gaus-Ostrogradski)



S je samo omotač kupe, dodaćemo osnovu kupe S_1, da bismo primijenili Gaus-Ostrogradskog

$$I = \iint\limits_{S \cup S_1} - \iint\limits_{S_1}$$

S_1: gornja strana

$n=(0,0,1)$

$z=h$ $\quad \varepsilon=1 \quad (x,y)\to(x,y,h)$

$(x,y)\in D \qquad D=\{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\le h^2\} \qquad \vec{n}=(0,0,1)$

$$I_1=\iint_{S_1} A\,ds=\iint_D h^2\,dx\,dy=h^2\iint_D dx\,dy=h^2\,\pi h^2=\pi h^4$$

$$I_2=\iint_{S\cup S_1} A=\iiint_{K} \underbrace{(2x+2y+2z)}_{\text{div}}\,dx\,dy\,dz = \text{uvodimo cil. koo.} =$$

K → konus

$$=\int_0^{2\pi}\int_0^{h}\int_r^{h}(r\cos\varphi+r\sin\varphi+z)\,r\,dz\,dr\,d\varphi=\dots=\frac{\pi h^4}{2}$$

$$I=I_2-I_1=\frac{h^4\pi}{2}-h^4\pi=-\frac{h^4\pi}{2}$$

9. Njutnovo polje

$$A(x)=\frac{x}{\|x\|^3}\quad,\ x\text{ - vektor sa 3 koordinate} \qquad \vec{A}(r)=\frac{\vec{r}}{\|r\|^3}$$

⇕

$$A(x,y,z)=\left(\frac{x}{(\sqrt{x^2+y^2+z^2})^3},\ \frac{y}{\dots},\ \frac{z}{\dots}\right)$$

→ u $(0,0,0)$ nije definisana

M^2 – proizv. zatv.

mnogostrukost koja:

a) ne sadrži O $(0,0,0)$ unutra

b) sadrži — // —

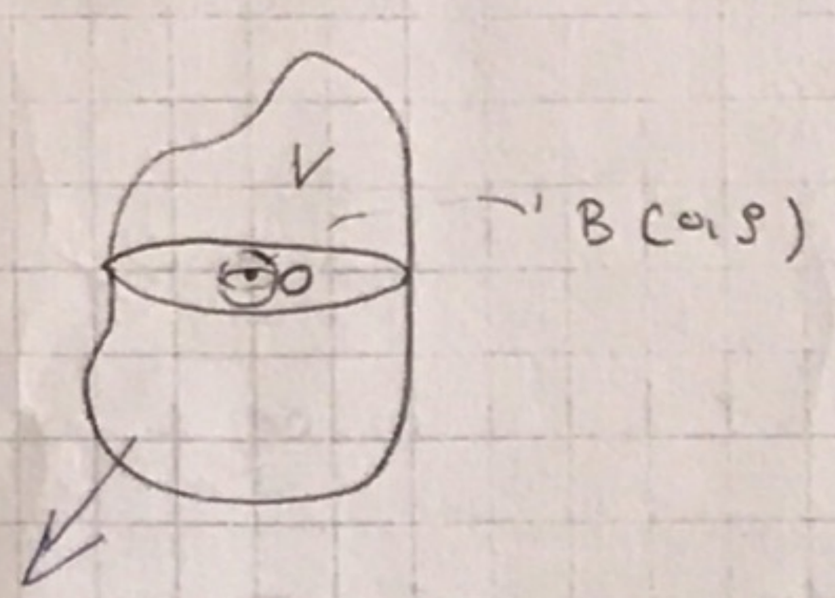
a) div $A=?$

$$\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{x^2+y^2+z^2}^{\,3}}\right)=\frac{y^2+z^2-2x^2}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

div $A=0$ $\qquad M^2=\partial V$

$$\int_{M_2} A=\iiint_V \text{div}\,A=\int_V 0=0$$

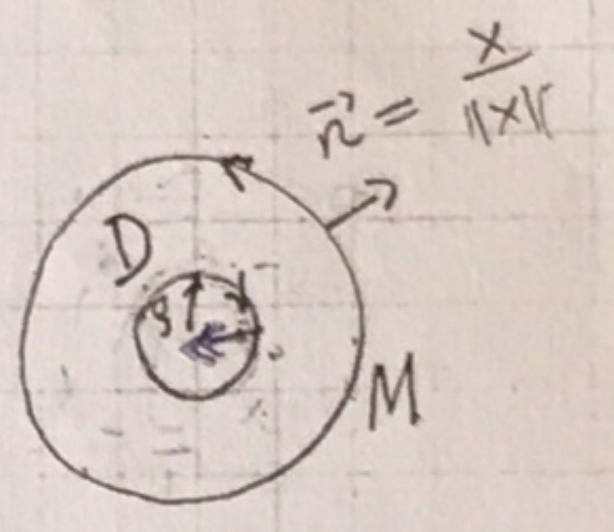
b) sadrži 0



ne možemo da primjenimo Gaus-Ostrogradskog jer sadrži 0 unutra ⇒ f-je nisu nepr. dif.
nisu čak ni definisane

$B(0,\rho) \subseteq V$

$D = V \setminus B(0,\rho)$ - V bez ⊖ nova oblast



$$\iint_{\partial D^+} A = \iiint_D \operatorname{div} A = 0 \quad (G-O)$$

$$\partial D = \overleftarrow{M}^2 + \overrightarrow{S}(0,\rho)$$

$$\underbrace{\iint_{\overleftarrow{M}^2} A}_{\text{pozitivno}} + \underbrace{\iint_{\overrightarrow{S}(0,\rho)} A}_{\text{negativno}} = 0 \Rightarrow \iint_{\overleftarrow{M}^2} A = \iint_{\overleftarrow{S}(0,\rho)} A = 4\pi$$

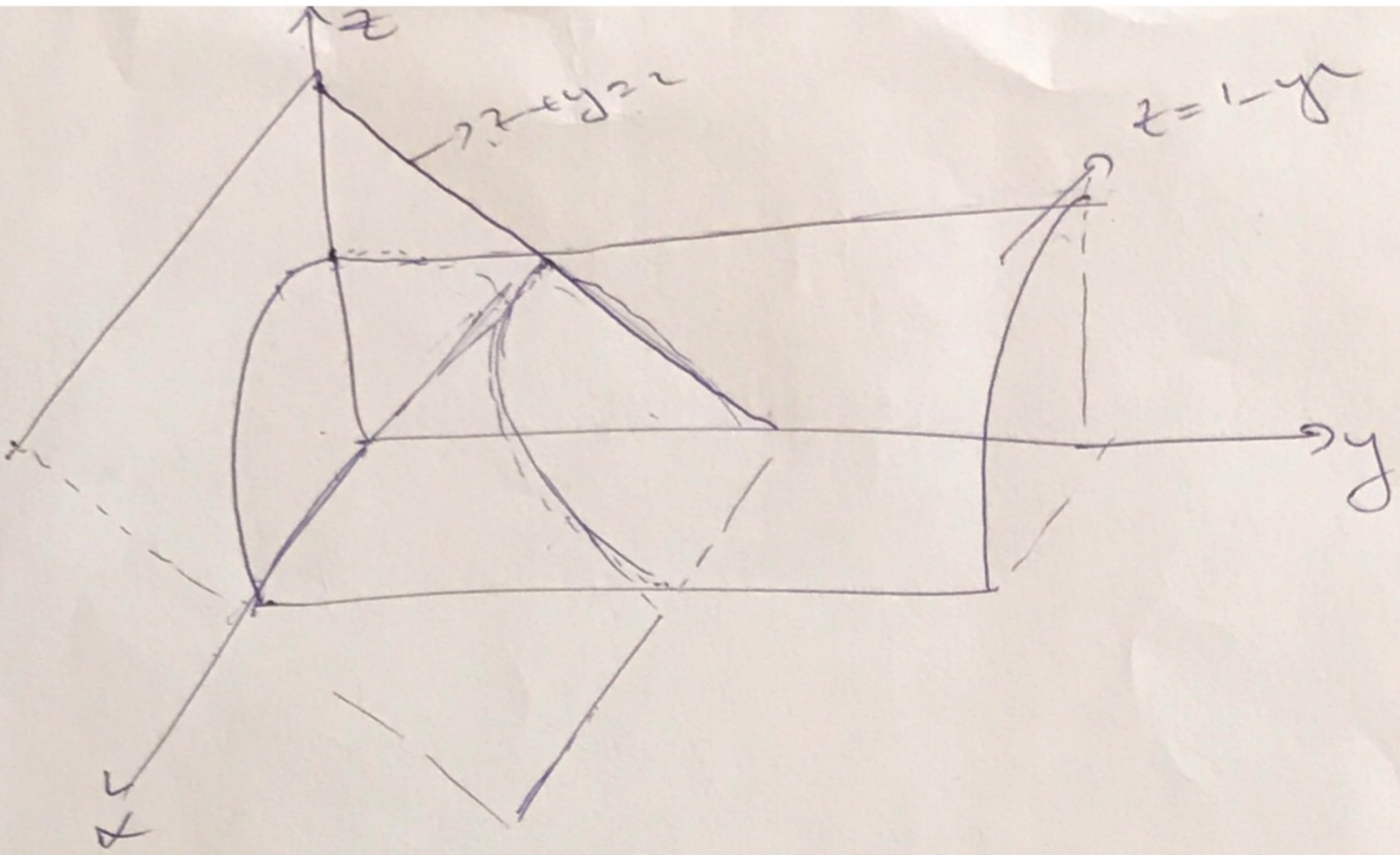
$$A(x) = \frac{x}{\|x\|^3}$$

jed. vektor normale

$$\int_{S^+(0,\rho)} A = \int_S \langle A, \vec{n}_x \rangle ds = \int_{S(0,\rho)} \langle \frac{x}{\|x\|^3}, \frac{x}{\|x\|} \rangle ds =$$

$$= \int_{S(0,\rho)} \frac{1}{\|x\|^2} = \frac{1}{\rho^2} \underbrace{\iint_S ds}_{4\pi\rho^2} = 4\pi$$

$x = \rho$


z
2z+y=2
z=1-y²
y
x